Valuing equity-linked death benefits

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Let T_x denote the time-until-death random variable for a life aged x.

Let $S(t)$ be the time-t price of a stock or mutual fund.

Let $M_s(t) := \max\{S(\tau), 0 \le \tau \le t\}$ be the maximum price of the stock or mutual fund up to time t.

Problem: Evaluate $E[e^{-\delta T_x} b(S(T_x), M_s(T_x))]$

where the expectation is taken with respect to an *appropriate* probability distribution and δ is a force of interest.

Examples:

$b(s,u) = (s-K)_+$ call option payoff

$b(s,u) = (K-s)_+$ put option payoff

$b(s,u) = u$ high water mark payoff

$b(s,u) = I(u < B)(K-s)_+$ up-and-out put

By conditioning,

$$E[e^{-\delta T_x} b(S(T_x), M_s(T_x))]$$

$$= E[E[e^{-\delta T_x} b(S(T_x), M_s(T_x)) \mid T_x]]$$

$$= \int_0^\infty E[e^{-\delta t} b(S(t), M_s(t)) \mid T_x = t] \, f_{T_x}(t) \, dt$$

$$= \int_0^\infty E[e^{-\delta t} b(S(t), M_s(t))] \, f_{T_x}(t) \, dt$$

if T_x is independent of $\{S(t)\}$.

So we want to calculate

$$\int_0^\infty E[e^{-\delta t} b(S(t), M_s(t))] f_{T_x}(t)\,dt.$$

If

$$f_{T_x}(t) = \sum_j c_j \, f_{\tau_j}(t) \ ,$$

then

$$\int_0^\infty E[e^{-\delta t} b(S(t), M_s(t))] f_{T_x}(t)\,dt.$$

$$= \sum_j c_j \int_0^\infty E[e^{-\delta t} b(S(t), M_s(t))] f_{\tau_j}(t)\,dt$$

$$= \sum_j c_j \, E[e^{-\delta \tau_j} b(S(\tau_j), M_s(\tau_j))].$$

Fact: The time-until-death density function can be *approximated* by linear combinations of *exponential* density functions

$$f_{T_x}(t) \approx \sum_j c_j \times f_{\tau_j}(t) = \sum_j c_j \times \lambda_j e^{-\lambda_j t}.$$

Thus, our valuation problem reduces to finding

$$E[e^{-\delta\tau} b(S(\tau), M_s(\tau))],$$

where τ is an *exponential* random variable

independent of $\{S(t)\}$.

Assume $S(t) = S(0)e^{X(t)}$, $t \geq 0$, where

$$X(t) = \mu t + \sigma Z(t) \quad + \sum_{j=1}^{N_\nu(t)} J_j - \sum_{k=1}^{N_\omega(t)} K_k$$

$$f_J(x) = \sum_{i=1}^{m} A_i v_i e^{-v_i x}, \quad x > 0; \quad E(N_\nu(t)) = \nu t$$

$$f_K(x) = \sum_{i=1}^{n} B_i w_i e^{-w_i x}, \quad x > 0; \quad E(N_\omega(t)) = \omega t$$

$$\sum_{i=1}^{m} A_i = 1, \quad \sum_{i=1}^{n} B_i = 1$$

Running maximum $M(t) := \mathrm{Max}\{X(u);\, 0 \le u \le t\}$

Because $S(t) = S(0)e^{X(t)}$,

$$M_s(t) = S(0)e^{M(t)}.$$

Thus, our valuation problem reduces to finding

$$E[e^{-\delta\tau} b(S(0)e^{x(\tau)}, S(0)e^{M(\tau)})],$$

where τ is an *exponential* random variable

independent of $\{S(t)\}$.

Fact: $M(\tau)$ and $[X(\tau) - M(\tau)]$ are independent random variables.

Thus,

$$E[e^{zX(\tau)}] = E[e^{z[X(\tau)-M(\tau)+M(\tau)]}]$$

$$= E[e^{z[X(\tau)-M(\tau)]}]\, E[e^{zM(\tau)}],$$

a version of *Wiener-Hopf factorization.*

Recall

$$X(t) = \mu t + \sigma Z(t) + \sum_{j=1}^{N_\nu(t)} J_j - \sum_{k=1}^{N_\omega(t)} K_k$$

where

$$f_J(x) = \sum_{i=1}^{m} A_i v_i e^{-v_i x}, \qquad x > 0$$

$$f_K(x) = \sum_{i=1}^{n} B_i w_i e^{-w_i x}, \quad x > 0$$

Then, $E[e^{zX(t)}] = e^{t\Psi(z)}$ for each $t \geq 0$, where

$$\Psi(z) = \mu z + \frac{1}{2}\sigma^2 z^2 + \nu \sum_{i=1}^{m} A_i \frac{z}{v_i - z} - \omega \sum_{i=1}^{n} B_i \frac{z}{w_i + z}$$

$$\Psi(z) = \mu z + \frac{1}{2}\sigma^2 z^2 + \nu\sum_{i=1}^{m} A_i \frac{z}{v_i - z} - \omega\sum_{i=1}^{n} B_i \frac{z}{w_i + z}$$

The moment-generating function of $X(\tau)$ is

$$E[e^{zX(\tau)}] = E[E[e^{zX(\tau)}|\tau]]$$

$$= E[e^{\tau\Psi(z)}] = \frac{\lambda}{\lambda - \Psi(z)}$$

The zeros of the RHS are the (m+n) poles of $\Psi(z)$,

namely, $v_1, \ldots, v_m, -w_1, \ldots, -w_n$.

The poles of the RHS are the (m+n+2) zeros of

$\lambda - \Psi(z)$.

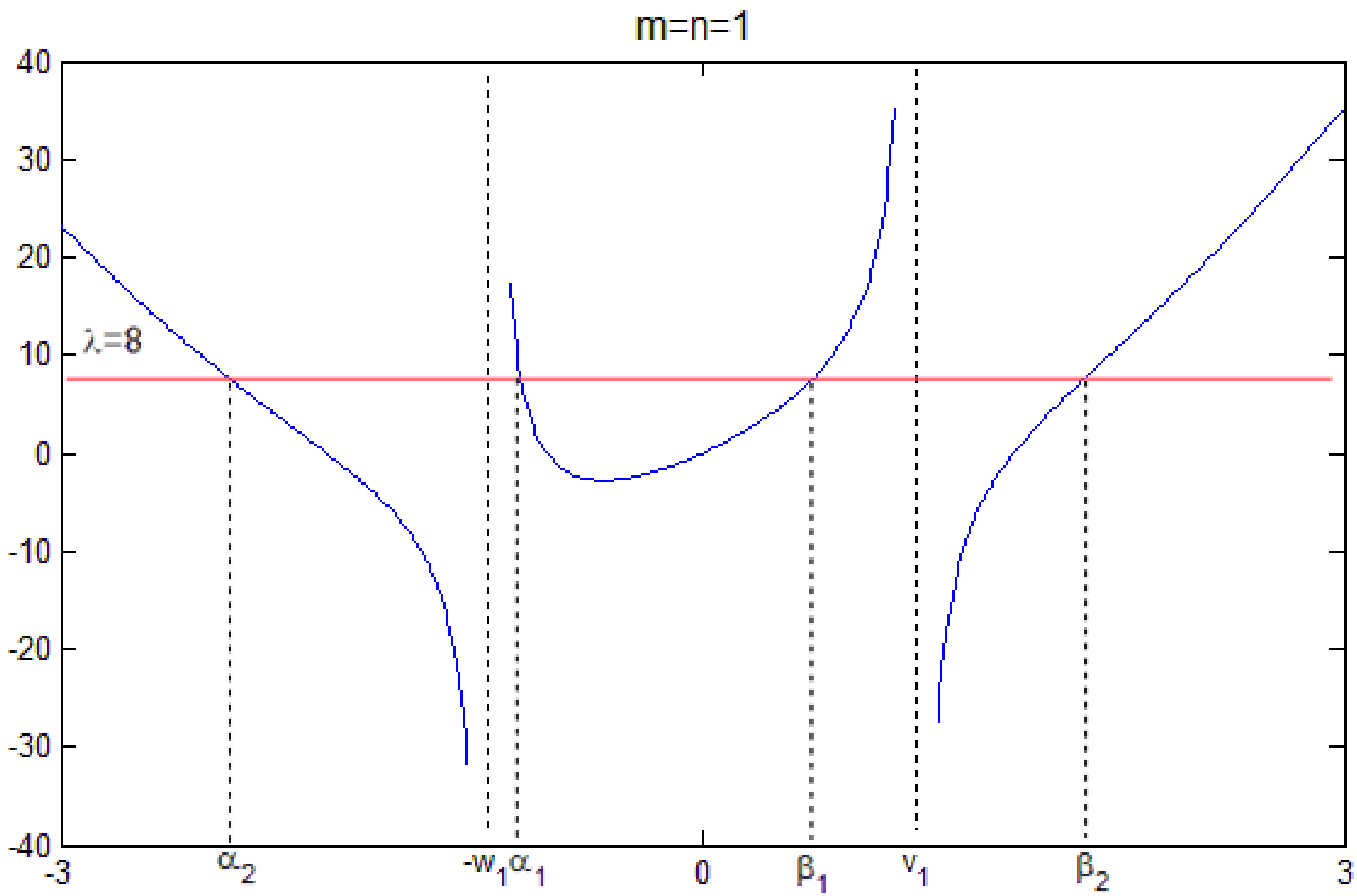
If $m = n = 1$,

$$\Psi(z) = \mu z + \frac{1}{2}\sigma^2 z^2 + \nu_1 \frac{z}{v_1 - z} - \omega_1 \frac{z}{w_1 + z}$$

$$E\left[e^{zX(\tau)}\right] = \frac{\lambda}{\lambda - \Psi(z)}.$$

The zeros are v_1 and $-w_1$, the poles of $\Psi(z)$.

The poles are the 4 zeros of $\lambda - \Psi(z)$.



$-w_1$, v_1, are the poles of $\Psi(z)$

$\alpha_2, \alpha_1, \beta_1, \beta_2$ are the zeros of $\lambda - \Psi(z)$

If $m = n = 2$,

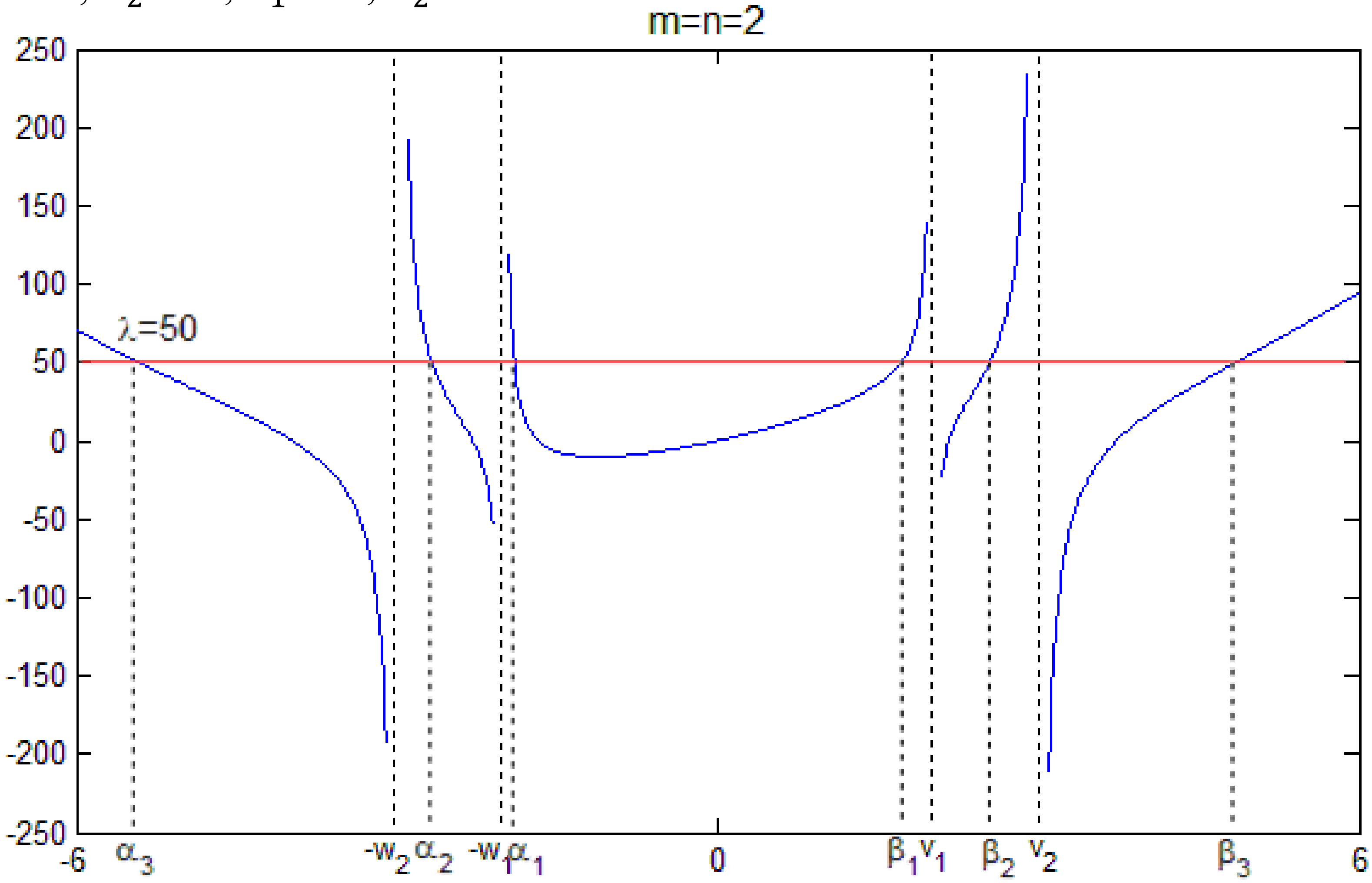
$$\Psi(z) = \mu z + \frac{1}{2}\sigma^2 z^2 + \nu\sum_{i=1}^{2} A_i \frac{z}{v_i - z} - \omega\sum_{i=1}^{2} B_i \frac{z}{w_i + z}$$

$$E\left[e^{zX(\tau)}\right] = \frac{\lambda}{\lambda - \Psi(z)}.$$

The zeros are $v_1, v_2, -w_1, -w_2$, the poles of $\Psi(z)$.

The poles are the 6 zeros of $\lambda - \Psi(z)$.

$A_1 > 0, A_2 > 0, B_1 > 0, B_2 > 0$



$-w_2, -w_1, v_1, v_2$ are the poles of $\Psi(z)$

$\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ are the zeros of $\lambda - \Psi(z)$

Label the parameters (the poles of $\Psi(z)$) such that

$$v_1 < v_2 < \ldots < v_m \quad ,$$

$$w_1 < w_2 < \ldots < w_n$$

If the weights A's and B's are positive, then the (m+n+2) zeros of $\lambda - \Psi(z)$ are interlaced as follows,

$$-\infty < \alpha_{n+1} < -w_n < \ldots < -w_1 < \alpha_1 < 0 < \beta_1 < v_1 < \ldots < v_m < \beta_{m+1} < \infty$$

$$E[e^{zX(\tau)}] = \frac{\lambda}{\lambda - \Psi(z)}$$

$$\propto \left(\prod_{j=1}^{m}(z - v_j)\right)\left(\prod_{j=1}^{n}(z - (-w_j))\right)\left(\prod_{j=1}^{m+1}\frac{1}{z - \beta_j}\right)\left(\prod_{j=1}^{n+1}\frac{1}{z - \alpha_j}\right)$$

$$E[e^{zX(\tau)}] \propto \left(\prod_{j=1}^{m}(z - v_j)\right)\left(\prod_{j=1}^{n}(z + w_j)\right)\left(\prod_{j=1}^{m+1}\frac{1}{z-\beta_j}\right)\left(\prod_{j=1}^{n+1}\frac{1}{z-\alpha_j}\right)$$

Wiener-Hopf factorization: $E[e^{zX(\tau)}] = E[e^{zM(\tau)}] \times E[e^{z[X(\tau)-M(\tau)]}]$

$$M(\tau) > 0 \quad \Rightarrow 0 < E[e^{zM(\tau)}] \le 1, \quad \text{for } -\infty < z < 0$$

$$[X(\tau) - M(\tau)] < 0 \quad \Rightarrow 0 < E[e^{z[X(\tau)-M(\tau)]}] \le 1, \quad \text{for } 0 < z < \infty$$

Hence,

$$E[e^{zM(\tau)}] \propto \left(\prod_{j=1}^{m}(z - v_j)\right)\left(\prod_{j=1}^{m+1}\frac{1}{z-\beta_j}\right).$$

$$E[e^{z[X(\tau)-M(\tau)]}] \propto \left(\prod_{j=1}^{n}(z + w_j)\right)\left(\prod_{j=1}^{n+1}\frac{1}{z-\alpha_j}\right).$$

$$E[e^{zM(\tau)}] \propto \left(\prod_{j=1}^{m}(z-v_j)\right)\left(\prod_{j=1}^{m+1}\frac{1}{z-\beta_j}\right).$$

Because $\quad E[e^{0M(\tau)}]=1,$

$$E[e^{zM(\tau)}]=\left(\prod_{j=1}^{m}\frac{v_j-z}{v_j}\right)\left(\prod_{j=1}^{m+1}\frac{\beta_j}{\beta_j-z}\right)$$

$$=\sum_{k=1}^{m+1}\left(\prod_{j=1}^{m}\frac{v_j-\beta_k}{v_j}\right)\left(\prod_{j=1,\,j\neq k}^{m+1}\frac{\beta_j}{\beta_j-\beta_k}\right)\frac{\beta_k}{\beta_k-z} \qquad \text{partial fractions}$$

$$=\sum_{k=1}^{m+1} b_k\frac{1}{\beta_k-z}, \qquad \text{where } b_k=\left(\prod_{j=1}^{m}\frac{v_j-\beta_k}{v_j}\right)\left(\prod_{j=1,\,j\neq k}^{m+1}\frac{\beta_j}{\beta_j-\beta_k}\right)\beta_k$$

Thus, $\quad f_{M(\tau)}(x)=\sum_{k=1}^{m+1} b_k e^{-\beta_k x}, \qquad x>0$

$$f_{M(\tau)}(x) = \sum_{k=1}^{m+1} b_k e^{-\beta_k x}, \qquad x > 0,$$

$$\text{where} \quad b_k = \left(\prod_{j=1}^{m} \frac{v_j - \beta_k}{v_j} \right) \left(\prod_{j=1, j \neq k}^{m+1} \frac{\beta_j}{\beta_j - \beta_k} \right) \beta_k$$

Similarly,

$$f_{X(\tau)-M(\tau)}(x) = \sum_{k=1}^{n+1} a_k e^{-\alpha_k x}, \qquad x < 0,$$

$$\text{where} \quad a_k = \left(\prod_{j=1}^{n} \frac{\alpha_k + w_j}{w_j} \right) \left(\prod_{j=1, j \neq k}^{n+1} \frac{-\alpha_j}{\alpha_k - \alpha_j} \right) (-\alpha_k)$$

For $y \geq 0$ and $y \geq x$,

$$f_{X(\tau),M(\tau)}(x,y)$$

$$= f_{M(\tau),X(\tau)-M(\tau)}(y, x-y) \times 1 \qquad 1 = |\det J|$$

$$= f_{M(\tau)}(y) \times f_{X(\tau)-M(\tau)}(x-y) \qquad \because \text{ Independence}$$

$$= \left(\sum_{k=1}^{m+1} b_k e^{-\beta_k y} \right) \left(\sum_{j=1}^{n+1} a_j e^{-\alpha_j (x-y)} \right)$$

$$= \sum_{k=1}^{m+1} \sum_{j=1}^{n+1} a_j b_k \times e^{-\alpha_j x} e^{-(\beta_k - \alpha_j) y}$$

$$-\infty < \alpha_{n+1} < -w_n < \ldots < -w_1 < \alpha_1 < 0 < \beta_1 < v_1 < \ldots < v_m < \beta_{m+1} < \infty$$

$$E[e^{-\delta\tau} b(S(\tau), M_s(\tau))]$$

$$= E[e^{-\delta\tau}] \times E^*[b(S(\tau), M_s(\tau))], \quad \text{where} \quad E^*[\tau] = \frac{1}{\delta + \lambda}.$$

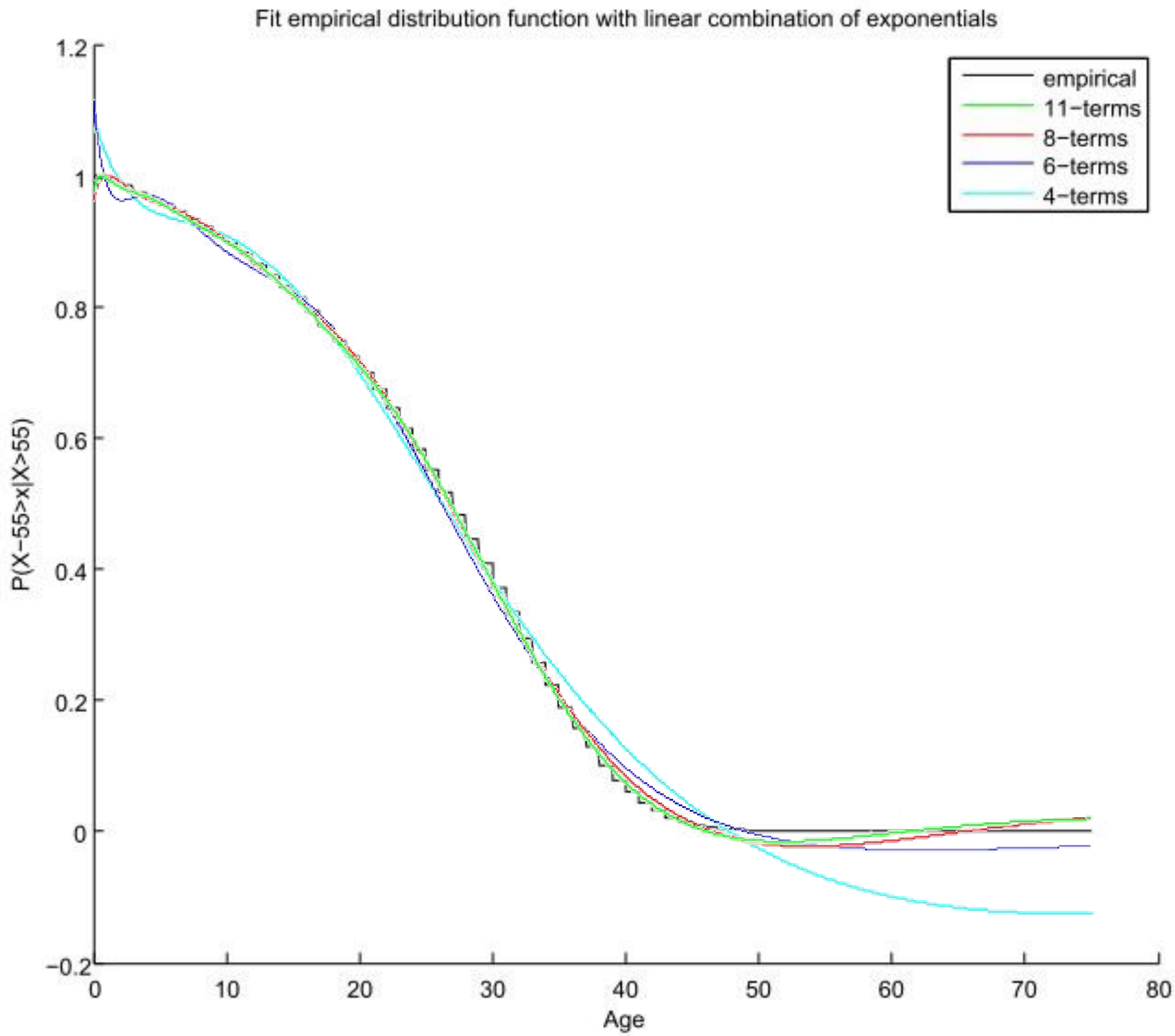
$$E^*[b(S(\tau), M_s(\tau))]$$

$$= E^*[b(S(0)e^{X(\tau)}, S(0)e^{M(\tau)})]$$

$$= \iint b(S(0)e^x, S(0)e^y) f^*{}_{X(\tau),M(\tau)}(x, y)\,dx\,dy$$

$$= \int_0^\infty \left[\int_{-\infty}^{y} b(S(0)e^x, S(0)e^y) f^*{}_{X(\tau),M(\tau)}(x, y)\,dx \right] dy$$

Numerical results



Numerical results

Stock parameter		ω	130
μ	0.05	ν	160
σ	0.2	A_1	1
w_1	5000	B_1	1
v_1	1666.6667	S(0)	1920

	$E(S(T_{55}))$	$E((1870-S(T_{55}))+)$	$E((S(T_{55})-1870)+)$	$E((1950-S(T_{55}))+)$	$E((S(T_{55})-1950)+)$
4 terms	2.08E+003	656.5394	712.8571	714.0115	683.5962
6 terms	1.92E+003	678.2119	731.1723	732.8594	705.7788
8 terms	1.85E+003	692.6016	744.0242	746.0782	720.5704
11 terms	1.88E+003	691.7957	743.8862	745.8406	719.8078

Thank You